Chapter 7: Techniques of Integration

Section 7.1: Integration by Parts

Objective: In this lesson, you learn

- ☐ how to obtain integration by parts as the rule corresponding to the Product Rule for differentiation and apply it to evaluate integrals.

I. Integration by Parts

Problem: Integrate

$$\int xe^x\,dx$$

What is the antiderivative of xe^x?

$\boxed{\ln e^x = x}$

1. let $u = x \rightarrow du = dx \Rightarrow \int u e^u\,du$!!

$u = e^x \Rightarrow \ln u = \cancel{\ln} e^x$

$\boxed{\ln u = x}$

2. let $u = e^x \rightarrow du = e^x dx \longrightarrow$

$\int x e^x\,dx = \int x\,du = \int \ln u\,du$!!!

Every differentiation rule has a corresponding integration rule. For example, the **Substitution Rule** for integration corresponds to the **Chain Rule for differentiation**.

The rule that corresponds to the **Product Rule for differentiation** is called the rule for **integration by parts.**

Recall that the Product Rule states that if $f(x)$ and $g(x)$ are differentiable functions, then

$$\int \cancel{\frac{d}{dx}} [f(x)g(x)] = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

In the notation for indefinite integrals, this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x)$$

Rearrange the equation to write the formula for **integration by parts**:

$$\boxed{\int \underbrace{f(x)}_{u}\underbrace{g'(x)dx}_{dv} = \underbrace{f(x)}_{u}\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\underbrace{f'(x)dx}_{du}}$$

In order to remember the formula more easily, let

$$\begin{aligned} u = f(x) \quad & \overset{D}{\Longrightarrow} \quad du = f'(x)dx, \\ dv = g'(x)dx \quad & \overset{\int}{\Longrightarrow} \quad v = g(x) \end{aligned}$$

So the formula above becomes:

$$\boxed{\int udv = uv - \int vdu}$$

Note that the purpose of using this formula is to obtain a simpler integral so choose u and v carefully so that the resulting integral is simpler than the original one.

Example 1: Integrate

$$\int xe^x\,dx$$

$$u = x \longrightarrow du = dx$$

$$dv = e^x\,dx \xrightarrow{\int} v = e^x$$

$$\int u\,dv = uv - \int v\,du$$

$$\int x e^x\,dx = x e^x - \int e^x\,dx$$

$$\boxed{\int x e^x\,dx = x e^x - e^x + C}$$

$$(x e^x - e^x)' = \frac{d}{dx}(x e^x) - \frac{d}{dx}(e^x)$$

$$= (x e^x + \cancel{e^x \cdot 1}) - (\cancel{e^x})$$

Choose u and v carefully!

Choose a u that gets simpler when you differentiate it and a v that doesn't get any more complicated when you integrate it.

A helpful rule of thumb is **I LATE**, choose u based on which of these comes first:

- I: Inverse trigonometric functions such as $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$.
- L: Logarithmic functions such as $\ln(x)$, $\log(x)$.
- A: Algebraic functions such as x^2, x^3.
- T: Trigonometric functions such as $\sin(x), \cos(x), \tan(x)$
- E: Exponential functions such as $e^x, 3^x$.

Example 2: Integrate

$$\int x^2 e^x dx$$

$$u = x^2 \xrightarrow{D} du = 2x\, dx$$

$$dv = e^x dx \xrightarrow{I} v = e^x$$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$

$$= x^2 e^x - 2\int x e^x dx$$

from Example 1.

$$= x^2 e^x - 2\cdot(x e^x - e^x) + C$$

$$= x^2 e^x - 2x e^x + 2e^x + C.$$

u (D)		dv (I)
x^2		e^x
$2x$	$+$	e^x
2	$-$	e^x
0	$+$	e^x

$$\int x^2 e^x = x^2 e^x - 2x e^x + 2e^x + C.$$

Example 3: Integrate

$$\int x^3 e^x dx$$

u (D)		dv (I)
		e^x
x^3	$+$	e^x
$3x^2$	$-$	e^x
$6x$	$+$	e^x
6	$-$	e^x
0		

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C.$$

Example 4: Integrate

$$\int x^2 e^{-x} dx$$

u (D)		dv (I)
		e^{-x}
x^2	$+$	$-e^{-x}$
$2x$	$-$	e^{-x}
2	$+$	$-e^{-x}$
0		

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

Example 5: Integrate

$$\int x \cos x dx$$

u (D)		dv (I)
x	$+$	$\cos x$
1	$-$	$-\sin x$
0		$-\cos x$

$$\int x \cos x \, dx = -x \sin x + \cos x + C$$

Example 6: Integrate

$$\int \ln x \, dx$$

$$u = \ln x \xrightarrow{D} du = \frac{1}{x} dx$$

$$dv = 1\, dx \xrightarrow{I} v = x$$

$$uv - \int v\, du$$

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$\boxed{\int \ln x \, dx = x \ln x - x + C}$$

$$(x \ln x - x)' = \frac{d}{dx}(x \ln x) - \frac{d}{dx}(x)$$

$$= \left[x \cdot \frac{1}{x} + \ln x \,(1)\right] - 1$$

$$= 1 + \ln x - 1 = \ln x$$

Example 7: Integrate

$$\int x^2 \ln x\, dx$$

$$I$$

② let $u = x^2 \xrightarrow{D} du = 2x\,dx$

$dv = \ln x\,dx \xrightarrow{I} v = x\ln x - x$

$uv - \int v\,du$

$$\int x^2 \ln x\,dx = x^2(x\ln x - x) - \int (x\ln x - x)(2x)\,dx$$

$$I = x^3\ln x - x^3 - \int 2x^2\ln x - 2x^2\,dx$$

$$= x^3\ln x - x^3 - 2\int x^2\ln x\,dx + 2\int x^2\,dx$$

$$I = x^3\ln x \boxed{-x^3} - 2I + \boxed{2\frac{x^3}{3}} + C$$

$+2I \qquad +2I$

$$3I = x^3\ln x + \left(-1+\frac{2}{3}\right)x^3 + C$$

$$3I = x^3\ln x - \frac{1}{3}x^3 + C$$

$$I = \frac{x^3\ln x}{3} - \frac{1}{9}x^3 + C.$$

① let $u = \ln x \xrightarrow{D} du = \frac{1}{x}\,dx$

$dv = x^2\,dx \xrightarrow{I} v = \frac{x^3}{3}$

$uv - \int v\,du$

$$I = \frac{x^3}{3}\ln x - \int \frac{x^{\cancel{3}\,2}}{3}\cdot\frac{1}{\cancel{x}}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\left(\frac{x^3}{3}\right) + C$$

$$= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C.$$

Example 8: Integrate

$$\int \arcsin x\, dx$$

$u = \arcsin x \xrightarrow{D} du = \frac{1}{\sqrt{1-x^2}}\,dx$

$dv = dx \xrightarrow{I} v = x$

$uv - \int v\,du$

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$\boxed{u = 1-x^2} \rightarrow du = -2x\,dx$

$\boxed{\frac{du}{-2} = x\,dx}$

$$= x\arcsin x - \int \frac{1}{\sqrt{u}}\,\frac{-du}{2}$$

$$= x\arcsin x + \frac{1}{2}\int u^{-1/2}\,du$$

$$= x\arcsin x + \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{1}u^{1/2} + C$$

$$= x\arcsin x + (1-x^2)^{1/2} + C.$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

Example 9: Integrate

$$\int e^{2x}\cos x\,dx \qquad I$$

$u = \cos x \longrightarrow du = -\sin x\,dx$

$dv = e^{2x}\,dx \longrightarrow v = \frac{1}{2}e^{2x}$

$$I = \frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx$$

$u = \sin x \xrightarrow{D} du = \cos x\,dx$

$dv = e^{2x}dx \xrightarrow{I} v = \frac{1}{2}e^{2x}$

$$I = \frac{1}{2}e^{2x}\cos x + \frac{1}{2}\left[\frac{1}{2}e^{2x}\sin x - \frac{1}{2}\underbrace{\int e^{2x}\cos x\,dx}_{I}\right]$$

$$I = \frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}\sin x - \frac{1}{4}I$$

$$+\frac{1}{4}I \qquad +\frac{1}{4}I$$

$$\frac{4}{5}\cdot\frac{5}{4}I = \frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}\sin x \qquad \frac{4}{5}$$

$$I = \frac{4}{5}\left(\frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}\sin x\right) + C$$

$$\int \tan^2 x\, dx$$
$$\boxed{1+\tan^2 x = \sec^2 x}$$
$$\int \sec^2 x - 1\, dx$$
$$\tan x - x + C$$

Example 10: Integrate

$$\int x \tan^2(x)\, dx$$

$$u = x \xrightarrow{D} du = dx$$
$$dv = \tan^2 x\, dx \xrightarrow{I} v = \tan x - x$$

$$-\int$$

$$uv - \int v\, du$$

$$\int x \tan^2 x\, dx = x(\tan x - x) - \int \tan x - x\, dx$$
$$= x\tan x - x^2 - \ln|\sec x| + \frac{x^2}{2} + C$$

Example 11: Integrate

$$\int x^3 \sin\left(x^2\right) dx$$

u-sub.

let $t = x^2 \rightarrow dt = 2x\, dx \rightarrow dx = dt/2x$

$$\int x^3 \sin(x^2)\, dx = \int x^{\cancel{3}\,2} \cdot \sin(t) \cdot \frac{dt}{2\cancel{x}}$$
$$= \frac{1}{2}\int x^2 \sin(t)\, dt$$
$$= \frac{1}{2}\int t \sin(t)\, dt.$$

D		I
t	$+$	$\sin t$
1	$-$	$\cos t$
0		$-\sin t$

$$= \frac{1}{2}\left(t\cos t + \sin t\right) + C$$

$$\int x^3 \sin(x^2)\, dx = \frac{1}{2}\left(x^2 \cos x^2 + \sin x^2\right) + C.$$

$$\frac{d}{dx}\tan^{-1}(f(x)) = \frac{1}{1+f(x)^2}\cdot f'(x)$$

Theorem

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx.$$

Example 12: Evaluate

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)\,dx$$

$u = \arctan(1/x) \xrightarrow{D} du = \frac{1}{1+(\frac{1}{x})^2}\cdot\frac{-1}{x^2}\,dx$

$dv = dx \xrightarrow{I} v = x \quad dx$

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx = x\,\arctan\left(\frac{1}{x}\right)\Big]_1^{\sqrt{3}} + \int x \cdot \frac{1}{1+\frac{1}{x^2}} \cdot \frac{1}{x^2}\,dx$$

$$= \left[\left(\sqrt{3}\,\arctan\left(\frac{1/2}{\sqrt{3}/2}\right)\right) - \left(\arctan(1)\right)\right] + \frac{1}{2}\int_1^{\sqrt{3}} \frac{2x}{x^2+1}\,dx$$

$$= \left(\sqrt{3}\cdot\left(\frac{\pi}{6}\right) - \frac{\pi}{4}\right) + \frac{1}{2}\ln|x^2+1|\Big]_1^{\sqrt{3}}$$

$$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln(4) - \ln(2))$$

$$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{2} + \frac{1}{2}\ln(2)$$

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